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Spin dynamics of electrons in a unidirectional, inhomogeneous magnetic field

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Abstract. We investigate the dynamics of electrons of opposite spin in an inhomogeneous but unidirectional magnetic field. We compare the quantum expectation values of wavefunctions obtained from the Dirac equation with the averages of classical ensembles. Both confirm our analytical predictions for a parameter regime in which the electrons are separated but have partially spatially overlapping probabilities.

1. Introduction

The interaction of charged particles with spin in inhomogeneous magnetic field configurations is of wide interest. Historically, the relevance of the spin-field-gradient force to the spatial dynamics of electrons moving in such fields has been the issue of considerable debate. In 1928 Pauli, using a line of reasoning attributed to Bohr, argued that the requirement $\nabla \cdot \boldsymbol{B} = 0$ results in a Lorentz force along the primary field direction that would completely overshadow the effect of the spin–field interaction. This debate culminated in the assertion of Pauli that 'it is impossible to observe the spin of the electron, separated fully from its orbital momentum, by means of experiments based on the concept of classical particle trajectories' [1].

A series of experiments by Dehmelt and co-workers [2] have called into question the generality of the Bohr/Pauli edict by studying the dynamics of a single electron of known spin isolated in a Penning trap. The interaction of spins with inhomogeneous magnetic fields has also been investigated in the context of the quantum measurement process. Martens and de Muynck [3] analysed a field of the form $\boldsymbol{B} \propto (-bx, 0, bz - a)$ in which they assume a dominant dipole nature (i.e. $a \gg bx$ or bz in the region of interest).

The complicated B-field generated by two parallel wires with opposite currents has been considered by Batelaan *et al* [4] to study the dynamics of longitudinally spin polarized electrons. Brillouin [5] had suggested that a similar situation could lead to spatial separation of the electron spins, related to the classic Stern Gerlach experiment [6], a possibility that was also dismissed by Pauli [1].

Our aim is to study the electron dynamics in a field configuration of the form $\boldsymbol{B} \propto (0, 0, x)$, which is the simplest inhomogeneous field that is physically realizable (i.e. has zero divergence). This field configuration is commonly used in many areas of plasma physics [7] and can be generated by a uniform current density in the $-y$-direction. In geophysics it is studied in conjunction with charged particle dynamics in the Earth's magnetosphere [8], in which one finds regions of magnetic field reversal in so-called

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neutral current sheets. However, this configuration has not been used to study spin–field interactions, even though its unidirectional nature implies an important consequence. The spin component S_z is conserved, which allows investigation of the dynamics without the complication of taking spin flips caused by the precession into account.

In this paper we investigate the dynamical relevance of the coupling between the spin projection along the field direction and the gradient of the field. For a mixture of spin-up and spin-down electrons, our investigation reveals a splitting mechanism that is unusual in that it involves in part a dynamic reduction in the transverse width of the electrons wavepacket due to the coupling between the spin and the field gradient.

2. Classical ensemble dynamics: analytical results

The semi-classical non-relativistic motion of an electron in an inhomogenous magnetic field can be approximated by the Pauli-type Hamilton function:

$$H(\boldsymbol{r},\boldsymbol{p}) = \frac{1}{2}\left(\boldsymbol{p}+\frac{1}{c}\boldsymbol{A}\right)^2 + \frac{1}{c}\boldsymbol{S}\cdot\boldsymbol{\nabla}\times\boldsymbol{A} \tag{2.1}$$

where $\boldsymbol{S}$ denotes the spin, $\boldsymbol{A}$ is the vector potential, and atomic units ($e=\hbar=m=1, c\approx 137$) are used. We will focus our analysis on the dynamics in an inhomogeneous field given by

$$\boldsymbol{\nabla}\times\boldsymbol{A} = B_0(0,0,x) \tag{2.2}$$

which can be derived from the vector potential $\boldsymbol{A}=\frac{B_0}{2}(0,x^2,0)$. The parameter B_0 has units of magnetic field strength per length.

Our goal is to study the time evolution classically and quantum mechanically and to investigate the possible spatial separation of spins $S_z=\pm\frac{1}{2}$ along the x-axis. To model the quantum mechanical state used below we have chosen an ensemble of classical particles that are initially distributed according to

$$P_{\pm}(\boldsymbol{r},\boldsymbol{p},\boldsymbol{S},t=0) = \frac{1}{(2\pi\,\Delta x\,\Delta p_x)^3}\exp\left[-\frac{1}{2}\frac{r^2}{(\Delta x)^2}\right]\exp\left[-\frac{1}{2}\frac{p^2}{(\Delta p_x)^2}\right]\delta\left(S_z\mp\frac{1}{2}\right) \tag{2.3}$$

which has been chosen to minimize the Heisenberg uncertainty product in each coordinate direction, $\Delta x\Delta p_x=\frac{1}{2}$, and similarly for y and z. We will discuss whether it is possible to find a suitable parameter regime with respect to the B-field strength, the interaction time, and the parameters characterizing the initial distribution that will lead to an optimal separation of electrons with opposite spins. The spin term in equation (2.1) $\frac{B_0}{c}S_z x$ suggests that electrons with spin $S_z=-\frac{1}{2}(S_z=\frac{1}{2})$ move initially in the positive (negative) x-direction. As a quantitative measure of the degree of separation along the x-axis between both ensembles P_+ and P_- we define a function for either $S_z=\frac{1}{2}$ or $S_z=-\frac{1}{2}$ particles

$$Q(t) \equiv \frac{2|\langle x(t)\rangle|}{\Delta x(t)} \tag{2.4}$$

that can be calculated directly from a single ensemble of equal-spin particles. In analogy to the corresponding quantum mechanical expectation values, the classical ensemble average $\langle x\rangle$ denotes the position of the distribution, and $\Delta x\equiv\sqrt{\langle x^2\rangle-\langle x\rangle^2}$ is its time-dependent width. The average separation between the spin-up and -down ensembles is $2|\langle x\rangle|$, where $\langle x\rangle$ must be calculated from an equal-spin ensemble. If it is larger than the spatial width of each distribution, we have $Q>1$ and the distributions can be considered separated. It is therefore sufficient to restrict our discussion below to an ensemble with, e.g. $S_z=-\frac{1}{2}$ particles.

We now discuss the details of the dynamics. Due to the quadratic dependence of the vector potential on the spatial coordinate, the Hamilton function (2.1) has a remarkable scaling property that simplifies our analysis significantly. It we introduce new scaled space and time variables denoted by upper case symbols according to

$$\boldsymbol{R} \equiv \left(\frac{B_0}{4c}\right)^{1/3} \boldsymbol{r} \tag{2.5a}$$

and

$$T \equiv \left(\frac{B_0}{4c}\right)^{2/3} t \tag{2.5b}$$

the dynamics does not depend on the strength of the B-field at all, i.e. $H(\boldsymbol{r}, \boldsymbol{p}) = g^{2/3}\tilde{H}(\boldsymbol{R}, \boldsymbol{P})$, where we denote the scaling parameter as $g \equiv B_0/(4c)$ and where

$$\tilde{H}(\boldsymbol{R}, \boldsymbol{P}) = \tfrac{1}{2}[\boldsymbol{P} + 2(0, X^2, 0)]^2 + 4S_z X. \tag{2.5c}$$

This scaling leaves the action variable invariant: $\Delta X \Delta P_X = \Delta x \Delta p_x$. An important result of this scale invariance is that the (dimensionless) function Q does not depend on the magnitude of the magnetic field strength B_0. For the remainder of the discussion in this section we use the scaled (but not unitless) variables $\boldsymbol{R}$ and T unless otherwise noted.

Obviously, $\tilde{H}$ does not depend on Y or Z and the canonical momenta P_Y and P_z are conserved during the dynamics. The Hamilton equations of motion in the three directions can be simplified to the form:

$$\frac{\mathrm{d}^2}{\mathrm{d}T^2} X = -\frac{\mathrm{d}}{\mathrm{d}X} V_{\pm}(X) \tag{2.6a}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}T^2} Y = 4X \frac{\mathrm{d}}{\mathrm{d}T} X \tag{2.6b}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}T^2} Z = 0 \tag{2.6c}$$

where we define

$$V_{\pm}(X) \equiv 2X^4 + 2P_Y X^2 \pm 2X \tag{2.7}$$

and the negative sign corresponds to the effective potential for a spin-down [$S_z = -\frac{1}{2}$] electron.

It is interesting to note that the motion along the X-direction does not depend on the motion in the Y- or Z-direction and can be derived from the potential of equation (2.7). If one inserts the Hamilton equation of motion $P_Y = \frac{\mathrm{d}Y}{\mathrm{d}T} - 2X^2$ into equation (2.6a), one recovers the expected Lorentz force $-4\frac{\mathrm{d}Y}{\mathrm{d}T}X$ and the spin-gradient force $-4S_z$ on the right-hand side. The effective potentials $V_{\pm}$ for the two possible orientations of the spin are shown in figure 1 for $P_Y = 0$. For $S_z = -\frac{1}{2}$ the minimum is at $X_- = 4^{-1/3} \approx 0.63$ and the depth of the potential is $|V_-(X_-)| \approx 0.95$. Note that the scaled energy must be multiplied by $g^{2/3}$ to give the energy in atomic units. In other words, the depth of the (unscaled) potential increases with increasing B-field strength. As a numerical example, we note that the well depth is 1 meV for $g = 2.2 \times 10^{-7}$, corresponding to a huge field gradient $B_0 = 5.4 \times 10^{11}$ T m^{-1}. The spacing between the minima of the two potentials in our scaled units is $X_- - X_+ \approx 1.3$, and the maximum expected spatial separation of the electrons in atomic units, $x_- - x_+ = 1.3g^{-1/3}$, decreases with increasing field strength. This is somewhat counterintuitive as one might expect the spin separation to increase because the spin coupling in equation (2.1) increases linearly with B_0. However, the location of the minima is also confined by the potential, which grows quartically for large $|X|$ due to the

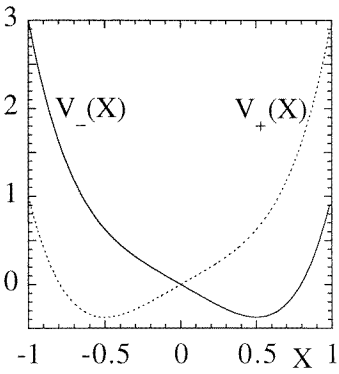

Figure 1. *Effective spin-gradient potential for spin-up and -down electrons*. The scaled potential functions $V_-(X)$ for spin-down electrons ($S_z = -\frac{1}{2}$) as defined in equation (2.7) for $P_Y = 0$. The dashed curve shows the corresponding potential $V_+(X)$ for $S_z = \frac{1}{2}$.

Lorentz force. Despite the nonlinearity, the equations (2.6) are fully integrable, and, for the special case of 'spin-free' charges, a quasianalytical solution can be derived†.

The term $2P_Y X^2$ in the potential reduces the depth of the minimum as well as its distance from $X = 0$. We focus our attention here on the conditions for an optimum spin separation and set $P_Y = 0$ for the remainder of the discussion.

Trivially, a single point particle located initially at $X = 0$ would move to the left or right depending on its spin, but because of the non-zero width due to the Heisenberg uncertainty relation, a more elaborate investigation using an ensemble is necessary. We will now present a simple analytical estimate that, in order to obtain the maximum possible spin separation, the spatial width of the initial ensemble should be in the range $0.4 < \Delta X < 1$. An effective separation between the two spins can be expected only if the average energy with respect to the X-direction in the initial ensemble is small. If the ensemble particles are too energetic the minima at $X_\pm$ become essentially irrelevant for their motion and the spin separation would become impossible. Let us assume that we can approximate the energy of the negative spin ensemble E_- as a function of the spatial width as $E_-(\Delta X) = (\Delta P_X)^2/2 + V_-(\Delta X)$. Furthermore, the minimum uncertainty product allows us to rewrite the kinetic energy in terms of the spatial variance via $(\Delta P_X)^2/2 = 1/(8\Delta X^2)$. One can show that this energy is negative ($E(\Delta X) < 0$) only if the width is restricted to the narrow range $0.4 < \Delta X < 1$. A smaller width leads to too large a kinetic energy and a larger width corresponding to more particles located at the sharply rising wings of the quartic potential leads to an undesired larger potential energy.

The prediction that the optimum initial width should be of the same order as the spacing between the two minima $X_- - X_+ \approx 1.3$ suggests that the distribution composed of, e.g. $S_z = -\frac{1}{2}$ electrons will initially reduce its width as its centre moves to the right. This reduction in width results from those ensemble particles (with $|X| > 0.6$) that are initially far from the origin where they experience an inward force directed towards the centre that acts against the natural spreading tendency. We will see below that both a decrease of the distribution width together with an increase in $\langle X \rangle$ will lead to an effective separation of both spins ($Q > 1$). We can also give a qualitative estimate for the time T^* required for the ensembles to drift toward the minima. Let us approximate this time by the time it takes particles that are initially located at $X = -\Delta X$ to move towards the minimum at X_-. As figure 1 shows, the slope of the potential $V_-(X)$ is nearly constant ($= 2$) for $|X| \leqslant 0.4$, and we can approximate the particle's motion towards X_- as uniformly accelerated. For $X = -0.6$ this time is about $T^* = 1.1$.

† A solution in terms of complementary incomplete elliptic integrals can be found in the second of [8].

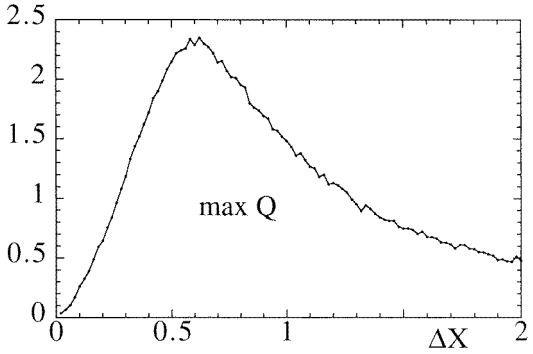

Figure 2. *Degree of spin separation.* Displayed is the maximum separation factor $Q = 2\langle X\rangle/\Delta X$ defined in equation (2.4) as a function of the initial spatial width ΔX of the ensemble P_-. The maximum Q was obtained from 20 000 trajectories that were integrated in time according to equation (2.6). For each of the 100 ensembles the maximum Q during the time evolution was monitored. The interaction time required to produce the largest Q was in almost all cases close to 1.1.

To test these analytical estimates for the optimum width and separation time and to obtain an estimate for the maximum achievable degree of separation Q that can be obtained in our B-field configuration, we have solved the equations of motion equation (2.6) numerically for 20 000 particles that were distributed according to P_- of equation (2.3). In figure 2 we display our main results. During the interaction time $0 < T < 2.6$ we have recorded the maximum separation via the factor $Q(T)$ defined in equation (2.4). We have repeated the simulations for 100 ensembles with various initial widths in the range $0 < \Delta X < 2$. The sharply rising wings of the quartic potentials $V_\pm$ constrain the transverse motion of the particles and thus restrict the maximum possible value for Q, i.e. an arbitrarily large degree of separation is not possible. The graph suggests that the optimum initial width is around $\Delta X \approx 0.6 \pm 0.3$ for which separation factors of $Q > 1.5$ can be achieved. This agrees well with our analytical estimate that this width should be in the range $0.4 < \Delta X < 1$. The corresponding maximum Q factor is close to 2.4. This numerical value is large enough to suggest that indeed it is possible to separate the spins, but the distributions overlap partially. The small fluctuations in the graph are expected for ensemble averages and scale with the inverse square root of the number of particles.

Using other data obtained from the same series of simulations we have also investigated a histogram of the distribution of those times T^* at which the Q factor took its maximum value, i.e. $Q_{\max}(T) = Q(T^*)$. The most probable time T^* was found to be close to 1.1 and the distribution around this value had a full width at half maximum of 0.1. This value for T^* agrees well with our analytical estimate.

To investigate the relative role of the initial momentum uncertainty and of the magnitude of the spin, we have repeated the simulations leading to figure 2 for ensembles whose spatial distribution was again given by equation (2.3), but whose initial momenta in the X-direction were sharp, i.e. $\Delta P_X = 0$. In this unphysical test case (which would obviously violate Heisenberg's uncertainty relation) the maximum Q decreases with increasing spatial width, but values of Q above 100 clearly show the importance of a small ΔP_X for an efficient separation. This finding is in agreement with another set of simulations for which $\Delta X \Delta P_X = 5$ that led to a maximum Q below 0.25. To investigate the scaling of the separation with the size of the spin, we have repeated the simulation of figure 2 with initial

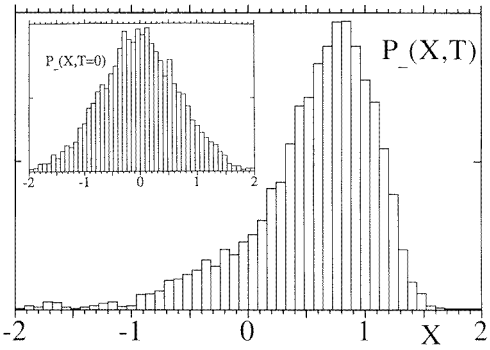

Figure 3. *Final spatial distribution.* We show the distribution of the position of 5000 particles with $S_z = -\frac{1}{2}$ at (scaled) time $T = 1.3$. The initial distribution of width $\Delta X = 0.7$ is shown in the inset.

spins that were (artificially) 10 times larger than permitted by nature, i.e. $S_z = -5$. The graph was similar to that obtained for $S_z = -\frac{1}{2}$, but with values of Q that were about six times larger.

In figure 3 we display the final spatial distribution $P_-(X, T)$ of the particles at $T = 1.3$ for 5000 spin-down particles. The corresponding distribution $P_+(X, T)$ for the spin-up particles is practically mirror symmetric, $P_+(X, T) = P_-(-X, T)$. The inset shows the initial distribution, which was chosen to have a variance of $\Delta X = 0.7$. The maximum separation factor for this case is $Q \approx 2.1$, as is evident from the graph in figure 2.

3. Quantum dynamics: numerical results

We now compare the results obtained from the classical ensembles with those obtained from quantum mechanical wavepackets. In order to investigate the quantum dynamics we have solved numerically the full time-dependent Dirac equation for an electron in the magnetic field described by equation (2.2):

$$\overleftrightarrow{H} = c\,\overleftrightarrow{\alpha}\left(\boldsymbol{p} + \frac{1}{c}\boldsymbol{A}\right) + c^2\,\overleftrightarrow{\beta} \tag{3.1}$$

where $\overleftrightarrow{\alpha}$ and $\overleftrightarrow{\beta}$ denote the well known 4×4 Dirac matrices [9]. It is well known that the Hamilton function (2.1) can be derived perturbatively from the Dirac Hamiltonian $\overleftrightarrow{H}$ if one omits terms of order $1/c^2$ and higher. As the initial state for our numerical solution we use the two spinor wavefunctions

$$\Psi^+(\boldsymbol{r}, t = 0) = (F(\boldsymbol{r}), 0, 0, 0) \tag{3.2a}$$

$$\Psi^-(\boldsymbol{r}, t = 0) = (0, F(\boldsymbol{r}), 0, 0) \tag{3.2b}$$

corresponding to states with a spin along the z-direction $\frac{1}{2}\left\langle \Psi^{\pm} \left| \begin{bmatrix} \sigma_z & 0 \\ 0 & \sigma_z \end{bmatrix} \right| \Psi^{\pm} \right\rangle = \pm\frac{1}{2}$, where σ_z denotes the Pauli 2×2 spin matrix. As a spatial wavefunction we have used

$$F(\boldsymbol{r}) = [2\pi(\Delta x)^2]^{-3/4} \exp[-(r/(2\Delta x))^2] \tag{3.3}$$

with an equal spatial uncertainty of Δx for each coordinate direction. This wavefunction leads to exactly the same moments in $\boldsymbol{r}$ and $\boldsymbol{p}$ as the classical ensemble distribution $P_\pm$ of equation (2.3). Because the spatial width is chosen orders of magnitude larger than the

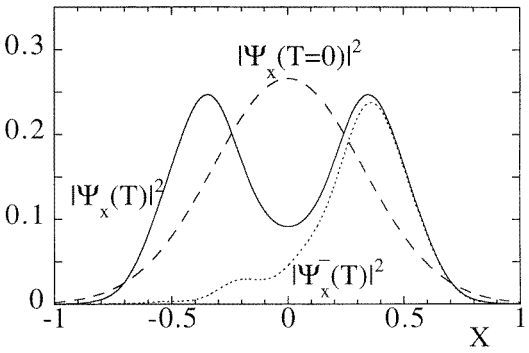

Figure 4. *Final spatial distribution of the wavefunction.* Final spatial probability of the quantum wavefunction Ψ (equation (3.5)) and Ψ^- (equation (3.2*b*)) (broken curve) at (scaled) time $T = 1.3$. The densities along the X-direction were obtained from the Dirac spinor wavefunction via equation (3.4). (The parameters used in the simulation were $g = 0.1$ au, $\Delta x = 1.5$ au, $t = 5.6$ au.)

Compton wavelength and we do not consider relativistic velocities, we can safely set the very small third and fourth spinor components in equation (3.2) equal to zero.

The Dirac equation is solved numerically on a discretized spacetime lattice using a generalized split-operator technique. In a typical simulation the product of the number of total grid points and the time step is on the order of 10^{10} and the calculation takes several days of CPU time on a fast multiprocessor computer. To establish the numerical accuracy we have doubled the number of time steps and also the number of grid points in each coordinate direction and found that the final wavefunctions were essentially unchanged. For details on the numerical algorithm we refer the reader to [10].

We now display the final spatial probability along the x-direction that is obtained from the fully time-evolved Dirac spinor wavefunction via

$$|\Psi_x(t)|^2 \equiv \int\int \mathrm{d}y\,\mathrm{d}z \sum_{i=1}^{4} |\Psi_i(x, y, z, t)|^2. \tag{3.4}$$

In figure 4 the broken curve corresponds to the spatial probability density for the initial spin superposition state

$$\Psi(\boldsymbol{r}, t = 0) = \frac{1}{\sqrt{2}}[\Psi^+(\boldsymbol{r}, t = 0) + \Psi^-(\boldsymbol{r}, t = 0)]. \tag{3.5}$$

The full curve shows the corresponding time-evolved two-peaked probability density $|\Psi_x(t)|^2$. The absence of any spatial interference structure between both spin contributions is, of course, expected as a result of the orthogonality of the spinors Ψ^+ and Ψ^-. The dotted curve shows the probability density obtained from the time-evolved spin-down state $\Psi^-(\boldsymbol{r}, t)$ from equation (3.2*b*). In agreement with our semiclassical results from section 2, the quantum case also predicts that it is possible to separate the spins, but again both distributions overlap partially. We should mention that the maximum quantum mechanical separation factor in this case is $Q = 2.6$, remarkably close to the value obtained from the classical ensembles.

In figure 5 we present a direct comparison between the first- and second-order moments obtained from the classical ensemble calculations P_- (dotted curves) and those from the quantum solutions Ψ^- of the Dirac equation. The classical ensemble average $\langle x(t)\rangle_{\mathrm{cl}}$ agrees remarkably well with the quantum expectation value $\langle x(t)\rangle$. The quantum excursion,

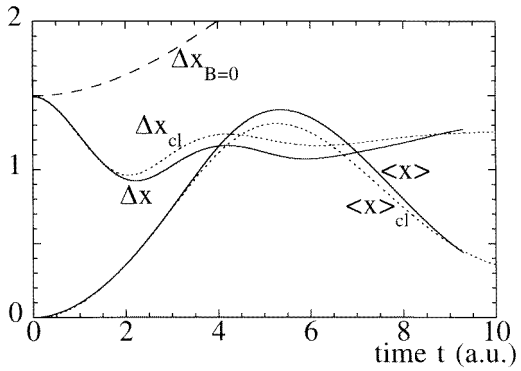

Figure 5. Direct comparison of the results for the quantum expectation values $\langle \ldots \rangle$ versus classical ensemble averages $\langle \ldots \rangle_{\rm cl}$ (dotted curves). For comparison, the broken curve at the top corresponds to the natural wavepacket spreading for an ensemble or wavefunction in the absence of any magnetic field ($B_0 = 0$).

however, is slightly larger than that of the classical ensemble and the width of the quantum wavepacket is slightly smaller than that of the ensemble. The largest relative difference between the quantum and the ensemble average value is only about 7%.

We have used atomic units in this figure to more easily relate the spatial variances to that of the well known force-free wavepacket spreading: $\Delta x(t)^2 = \Delta x^2 + \Delta p^2 t^2$. The broken curve in the upper left corner of the figure displays this growth of spatial width of a quantum wavepacket in the absence of any magnetic field ($B_0 = 0$). This spreading is, of course, identical to that of the corresponding force-free classical ensemble $\Delta x(t) = \Delta x(t)_{\rm cl}$ [11].

4. Discussion

We now summarize our primary results. These results were guided by simple analytical estimates derived from dynamics in a one-dimensional effective potential and agree with both numerical simulations of classical particle ensembles and fully *ab initio* solutions of the Dirac equation.

(1) Spatial separation transverse to the spin projection axis for both ensembles and quantum wavepackets is found for the magnetic field configuration given by equation (2.2). The degree of separation, described by $Q(t)$ from equation (2.4), is independent of the magnetic field strength (or, more precisely, the parameter B_0 describing the field gradient in equation (2.2)). Only the time required to achieve maximum separation is dependent on B_0.

(2) The maximum achievable separation for this field configuration is roughly twice the initial width of the classical or quantum distribution. There is an unavoidable partial overlap of the final spin distributions that has its origin in the initial momentum spread according to the uncertainty principle.

(3) The initial spatial width of the wavepacket or ensemble required for optimal separation and the required interaction time are uniquely determined by the parameter B_0 in equation (2.2). In terms of the scaling factor $g = B_0/4c$, we have for the initial spread $\Delta x \cong 0.6 g^{-1/3}$ and for the required interaction time for maximum separation $t^* \cong 1.1 g^{-2/3}$.

To aid the reader with conversion from the scaled units to SI units, we give the following

expressions for the required B_0 and t^* in terms of the chosen initial spread Δx:

$$B_0 = \frac{7.8 \times 10^{-14}\ \mathrm{T\ m^2}}{(\Delta x)^3} \tag{4.1a}$$

$$t^* = 2.6 \times 10^4 \frac{s}{m^2}(\Delta x)^2. \tag{4.1b}$$

A general remark on the direct comparison of classical and quantum expectation values is in order. It is interesting to note that the spin-up and -down superposition state in the z-direction equation (3.5) used in our quantum analysis is identical to an eigenstate of the spin operator in the x-direction: $\frac{1}{\sqrt{2}}(|+,z\rangle + |-,z\rangle) = |+,x\rangle$. An individual classical particle with an initial spin in the x-direction would not couple to the gradient force at all, as its time-dependent spin $\boldsymbol{S}(t)$ remains confined to the (S_x, S_y) plane in our unidirectional B-field along the z-axis. In our numerical analysis we have chosen the classical ensemble to consist of equal numbers of particles with $S_z = \frac{1}{2}$ and $S_z = -\frac{1}{2}$. This ensemble, however, has a zero ensemble average value $\langle S_x\rangle_{\mathrm{cl}} = 0$ and not $\langle S_x\rangle = \frac{1}{2}$ as for the quantum superposition state. To find a better match with respect to the spin along the x-axis as well, one would have to construct a classical ensemble from individual particles for which $\langle S_x\rangle_{\mathrm{cl}} = \frac{1}{2}$ and $\langle S_y\rangle_{\mathrm{cl}} = \langle S_z\rangle_{\mathrm{cl}} = 0$ and also for which $\langle {S_x}^2\rangle_{\mathrm{cl}} = \langle {S_y}^2\rangle_{\mathrm{cl}} = \langle {S_z}^2\rangle_{\mathrm{cl}} = \frac{1}{4}$ as required by the corresponding expectation values for the quantum wavefunction. For an arbitrary B-field configuration, we have not been able to construct a classical ensemble of spin particles that can satisfy all initial conditions and remain inside the (quantum mechanical) range $-\frac{1}{2} \leqslant \langle S_i(t)\rangle_{\mathrm{cl}} \leqslant \frac{1}{2}$ for $i = x, y, z$ during the entire time evolution. This reflects, of course, the well known fact that the spin dynamics cannot be described completely in terms of classical ensembles; there is after all something intrinsically quantum mechanical about spin [12]. In view of this it is remarkable that our analytical estimates based on the simple particle mechanics including spin agree with the quantum results. This good agreement could be due to the conservation of the z-projection of the spin for our B-field configuration.

We should also mention that an interpretation of the electron dynamics in terms of the usual Landau states [13] (stationary states in homogeneous fields) is difficult for our field configuration due to the field reversal in the $x = 0$ plane. We have not been able to find energy eigenstates of the Dirac operator for inhomogeneous magnetic fields. These generalized Landau states would be more suitable for our situation.

To summarize, we have identified a window in the parameter regime for which it is possible to separate spins using an inhomogeneous magnetic field. The degree of separation depends on the Heisenburg uncertainty product of the initial distribution in space and momentum with respect to the magnetic field's gradient direction. The degree of separation is enhanced if the uncertainty product in space and momentum along the field's gradient direction is minimal in the initial distribution.

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[10] For technical details of the numerical algorithm, see Braun J W, Su Q and Grobe R *Phys. Rev.* A in press
[11] For a detailed discussion of this analogy, see Su Q, Smetanko B A and Grobe R 1998 *Laser Phys.* **8**
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[12] The classical limit of spin-systems has been thoroughly analysed in quantum chaos. For a review, see Haake F 1991 *Quantum Signatures of Classical Chaos* (Berlin: Springer) and many references therein
[13] For an overview of Landau levels, see Sokolov A A and Temov I M 1968 *Synchrotron Radiation* (Berlin: Akademie)
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